1. (50 points) Let $N^{[2]} = N \times N$, $N^{[3]} = N \times (N \times N)$ and so forth. So for example, $D_{n,d} = X^{[k]}_d$ if $k = \log_d n$. Now, let

$$H(n, d_1, d_2, h, r) = \begin{cases} X^{[k]}_{d_1,d_1} \otimes X^{[k-h]}_{d_1,d_1} \otimes (X^{[k-r]}_{d_1,d_1} \times X^{[r]}_{d_1,d_2}) & \text{if } r \leq h \\ X^{[k]}_{d_1,d_1} \otimes (X^{[k-r]}_{d_1,d_1} \times X^{[r-h]}_{d_1,d_2}) \otimes X^{[h]}_{d_1,d_1} & \text{if } r \geq h \end{cases}$$

Note that if $d_1 = d_2$ that $H(n, d_1, d_2, h, r) \approx D^*_{n,d_1,h}$. Draw pictures of $H_{16,2,3,2,1}$ and $H_{16,2,3,2,2}$. 
Suppose this network is used to implement a multicast ATM switch with dynamic routing (the first $h$ stages distribute traffic, the remaining $k$ stages route and copy cells to the outputs). Let $\lambda_t(c_j)$ be the load that a virtual circuit $(x_j, Y_j, \omega_j)$ places on a link $\ell$ where $x_j$ is an input, $Y_j$ is a set of outputs and $\omega_j$ is a weight between zero and 1, representing the bandwidth of the virtual circuit’s data stream, relative to the rate of the internal links. If $\ell$ is in stage $i$ and is on some path from $x$ to one or more outputs in $Y$, what is the load that $c_j$ induces on $\ell$ if $i \leq h$?

$\omega_j d_1^{-i}$

What if $i > h$?

$\omega_j d_1^{-h}$

How many inputs can reach link $\ell$ if $i \leq h$?

$d_1^i$

What if $i > h$?

$n$

How many outputs can be reached from link $\ell$ if $i \leq h$?

$n(d_2/d_1)^r$

What if $i > h$?

$d_2^{h+k-i}$ if $r \geq h + k - i$, $d_2^{h+k-i-r} d_2^r$ otherwise.

What if $i < h$?

$d_2^{h+k-i}$ if $r \geq h + k - i$, $d_2^{h+k-i-r} d_2^r$ otherwise.

If the total traffic on each input and each output is $\leq \beta$, what is the maximum load that can be placed on link $\ell$ if $i \leq h$?

$\beta$

What if $i > h$?

$\beta(d_2/d_1)^{h+k-i}$ if $r \geq h + k - i$, $\beta(d_2/d_1)^r$ otherwise.

What if $i < h$?

$\beta d_1^{-h} \min \{d_1^i, d_2^{h+k-i}\}$ if $r \geq h + k - i$, $\beta d_1^{-h} \min \{d_1^i, d_2^{h+k-i-r} d_2^r\}$ otherwise.

What speed advantage is required to make this network nonblocking?

If $r \geq h + k - i$, the required speed advantage is

$$d_1^{-h} \max_i \{\min \{d_1^i, d_2^{h+k-i}\}\} = d_1^{-h + [(h+k) \log d_2 / \log d_1]}$$

Otherwise, it is

$$d_1^{-h} \max_i \{\min \{d_1^i, d_2^{h+k-i-r} d_2^r\}\} = d_1^{-h + [(h+k-r) \log d_1 + r \log d_2 / 2 \log d_1]}$$
2. (50 points) The WUGS switch includes a range-copy mechanism that can be used to multicast a cell to a set of consecutive outputs in a single pass through its network. We can use this to implement an arbitrary multicast virtual circuit addressed to \(2f\) different outputs, by first copying a cell to \(f\) consecutive ports, then recycling the copied cells back to the inputs and doing a VXT lookup on each copy, to get a pair of output ports and VXIs. The resulting cells are then sent into the switching network again, using the standard binary copy mechanism. This allows any multicast to be done in just two passes through the network, rather than requiring \(\log_2 f\) passes.

Suppose we want to set up a new multicast virtual circuit with fanout \(2f\) and bandwidth \(B\). How should we select a consecutive range of recycling ports to accommodate this new virtual circuit?

We need to find a consecutive range of recycling ports, all of which have a free VXT entry with the same index, and which all have \(\geq B\) units of bandwidth available.

If the multicast traffic can be as much as 20\% of the total outgoing traffic, what is the total traffic on all the recycling paths, assuming that there are \(n\) outgoing links with a load of a most \(\beta\) each and each port is shared by external traffic and recycling traffic?

The total traffic on the recycling ports is at most \(0.1\beta n\).

What condition must be satisfied to guarantee that we can always add a new virtual circuit with fanout \(2f\) and bandwidth \(B\)? (You may assume that \(f\) divides evenly into \(n\).)

In the worst case, we can block a new connection setup, so long as one recycling port in every \(f\) is has a load of at least \(1 - B\). So to prevent blocking, we need \(\beta + (0.1\beta n/(n/f)) + B \leq 1\) or \((1 + 0.1f)\beta + B \leq 1\).

What does this imply about the speed advantage that the system requires in order to be nonblocking?

\[1/\beta \geq 1 + 0.1f + (B/\beta)\]

If \((\beta/B) = 16\), what is the actual value of the required speed advantage for virtual circuits with fanout 4, 16 and 64? How does this compare with the case where we use binary copying and allow the number of recycling passes to increase with the fanout?

Since the virtual circuit fanout is \(2f\), we get \(1 + 0.2 + 0.06 = 1.26\), \(1 + 0.8 + 0.06 = 1.86\) and \(1 + 3.2 + 0.06 = 4.26\) in these three cases. Using binary copying and an unbounded number of recycling passes, the speed advantage can be just 1.26 in all cases.

How would the results above change, if the load on different recycling paths were guaranteed to differ by at most some value \(\Delta\)?

As before, a connection would block, only if one in every \(f\) recycling paths had a load of at least \(1 - B\), but now, the other recycling paths would have to have a load of at least \(1 - B - \Delta\) in this case. This changes the condition that must be satisfied to avoid blocking to

\[(n/f)(1 - \beta - B) + (n - n/f)(1 - \beta - B - \Delta) \geq 0.1n\]

Solving this for \(1/\beta\) gives

\[(1/beta) \geq (1.1 + (B/\beta) + (1 - 1/f)(\Delta/\beta))\]

If \(\beta/B = 16\) and the fanout is 4, the required speed advantage is \(1.16 + (1/2)(\Delta/\beta)\). For fanout 16, it is \(1.16 + (7/8)(\Delta/\beta)\). For fanout 64, it is \(1.16 + (31/32)(\Delta/\beta)\). So if \(\Delta/\beta\) is small, the required speed advantage can be only a little larger than if we use binary copying and an unlimited number of recycling passes.