Announcements

• Homework 1 is due Thursday

• Quiz 1 is next Tuesday

• Read Section 1.6 (Rules of Inference), 1.7 (Introduction to Proofs) and 1.8 (Proof Methods and Strategy by Thursday

Predicates

Alicia eats pizza at least once a week.

Define:

$EP(x) = \text{“} x \text{ eats pizza at least once a week.”}$

Universe of Discourse - $x$ is a student in cs240

A *predicate*, or propositional function, is a function that takes some variable(s) as arguments and returns True or False.

Note that $EP(x)$ is not a proposition, $EP(Ariel)$ is.
Another way of changing a predicate into a proposition.

Suppose $P(x)$ is a predicate on some universe of discourse.
Ex. $B(x) = "x$ is carrying a backpack," $x$ is set of cs240 students.

The universal quantifier of $P(x)$ is the proposition:
“$P(x)$ is true for all $x$ in the universe of discourse.”

We write it $\forall x \ P(x)$, and say “for all $x$, $P(x)$”

$\forall x \ P(x)$ is TRUE if $P(x)$ is true for every single $x$.
$\forall x \ P(x)$ is FALSE if there is an $x$ for which $P(x)$ is false.

Are either of these propositions true?

a) $\forall x \ (B(x) \rightarrow Y(x))$

b) $\forall x \ (Y(x) \lor L(x))$

What does this proposition mean?
$\forall x \ (Y(x) \land B(x))$

Is it true?

A: only $a$ is true
B: only $b$ is true
C: both are true
D: neither is true
Existential Quantifier

Another way of changing a predicate into a proposition.

Suppose $P(x)$ is a predicate on some universe of discourse.

Ex. $C(x) = “x \text{ has a candy bar,}” x \text{ is the set of cs240 students.}$

The existential quantifier of $P(x)$ is the proposition:

“$P(x)$ is true for some $x$ in the universe of discourse.”

We write it $\exists x \ P(x)$, and say “for some $x$, $P(x)$”

$\exists x P(x)$ is TRUE if there is an $x$ for which $P(x)$ is true.

$\exists x P(x)$ is FALSE if $P(x)$ is false for every single $x$.

$\exists x C(x)$?

What is this asking?

Is it true or false?

Existential Quantifier

$B(x) = “x \text{ is enrolled in CS240.”}$

$L(x) = “x \text{ is signed up for at least one class.”}$

Are either of these propositions true?

a) $\exists x \ B(x)$

b) $\exists x \ (B(x) \land L(x))$

What does this proposition mean?

$\exists x \ (B(x) \rightarrow L(x))$

Why do I have to be careful with this proposition?

A: only a is true
B: only b is true
C: both are true
D: neither is true
Predicate Examples

L(x) = “x is a lion.”
F(x) = “x is fierce.”
C(x) = “x drinks coffee.”

All lions are fierce.
∃x (L(x) ∧ ¬C(x))

Some lions don’t drink coffee.
∃x (L(x) ∧ ¬C(x))

Some fierce creatures don’t drink coffee.
∃x (F(x) ∧ ¬C(x))

More Examples

B(x) = “x is a hummingbird.”
L(x) = “x is a large bird.”
H(x) = “x lives on honey.”
R(x) = “x is richly colored.”

All hummingbirds are richly colored.
∀x (B(x) → R(x))

No large birds live on honey.
¬∃x (L(x) ∧ H(x))

Birds that do not live on honey are dully colored.
∀x (¬H(x) → ¬R(x))
More Examples

Universe of discourse is all people.

\[
S(x) = \text{“}x\text{ is a student in this class.”}
\]
\[
C(x) = \text{“}x\text{ has visited Chicago.”}
\]

∀x (S(x) ∧ C(x)) All people are students in this class and all people have visited Chicago

∀x (S(x) → C(x)) All students in this class have visited Chicago

∃x (S(x) → C(x)) Some student in this class has visited Chicago

∃x (S(x) ∧ C(x))

Quantifier Negation

Not all large birds live on honey.

\[\neg \forall x \ (L(x) \rightarrow H(x))\]

∀x P(x) means “P(x) is true for every x.”

What about \[\neg \forall x \ P(x)\] ?

Not \[\text{“}P(x)\text{ is true for every x.”}\]

“There is an x for which P(x) is not true.”

∃x \neg P(x)

So, \[\neg \forall x \ P(x)\] is the same as \[\exists x \neg P(x)\].

∃x \neg (L(x) \rightarrow H(x))
Quantifier Negation

No large birds live on honey.

∃x P(x) means “P(x) is true for some x.”

What about ¬∃x P(x)?

Not [“P(x) is true for some x.”]  
“P(x) is not true for all x.”

∀x ¬P(x)

So, ¬∃x P(x) is the same as ∀x ¬P(x).

∀x ¬(L(x) ∧ H(x))

Quantifier Negation

So, ¬∀x P(x) is the same as ∃x ¬P(x).

So, ¬∃x P(x) is the same as ∀x ¬P(x).

General rule: to negate a quantifier, move negation to the right, changing quantifiers as you go.
### Predicates – Multiple Quantifiers

<table>
<thead>
<tr>
<th>Quantifier</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \forall y P(x,y)$</td>
<td>$P(x,y)$ true for all $x$, $y$ pairs.</td>
</tr>
<tr>
<td>$\exists x \exists y P(x,y)$</td>
<td>$P(x,y)$ true for at least one $x$, $y$ pair.</td>
</tr>
<tr>
<td>$\forall x \exists y P(x,y)$</td>
<td>For every value of $x$ we can find a (possibly different) $y$ so that $P(x,y)$ is true.</td>
</tr>
<tr>
<td>$\exists x \forall y P(x,y)$</td>
<td>There is at least one $x$ for which $P(x,y)$ is true for every $y$.</td>
</tr>
</tbody>
</table>

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### Predicates – multiple quantifiers

Universe of discourse is all students in this room.

<table>
<thead>
<tr>
<th>Quantifier</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(x,y) = “x$ is sitting by $y”$</td>
<td>Universe of discourse is all students in this room.</td>
</tr>
<tr>
<td>$\forall x \forall y N(x,y)$</td>
<td>False</td>
</tr>
<tr>
<td>$\exists x \exists y N(x,y)$</td>
<td>True</td>
</tr>
<tr>
<td>$\forall x \exists y N(x,y)$</td>
<td>True</td>
</tr>
<tr>
<td>$\exists x \forall y N(x,y)$</td>
<td>False</td>
</tr>
</tbody>
</table>

Let “sitting by” be defined as $x$ is sitting within 5 feet of $y$. 
A **theorem** is a statement that can be shown to be true.

A **proof** is the means of doing so.

The following statements are true:

If I am Mila, then I am a great swimmer.
I am Mila.

What do we know to be true?
I am a great swimmer!

What rule of inference can we use to justify it?
Proofs - Definitions

• Argument
  – A sequence of propositions
  – All but the final proposition are called premises
  • Final proposition called conclusion
  – An argument is valid if the truth of all premises implies the conclusion is true

• Argument form
  – Sequence of compound propositions involving proposition variables

<table>
<thead>
<tr>
<th>premise</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>premise</td>
<td>( p \rightarrow q )</td>
</tr>
<tr>
<td>conclusion</td>
<td>( \therefore q )</td>
</tr>
</tbody>
</table>

Proofs - Modus Ponens

I am Mila.
If I am Mila, then I am a great swimmer.

\[ \therefore \text{I am a great swimmer!} \]

\[ \begin{align*}
  &p \\
  &p \rightarrow q \\
  \therefore &q
\end{align*} \]

Inference Rule: Modus Ponens

Tautology:

\[ (p \land (p \rightarrow q)) \rightarrow q \]

Modus Ponens is Latin for "the way that affirms by affirming"
I am not a great skater.
If I am Erik, then I am a great skater.

∴ I am not Erik!

\[ \neg q \]
\[ p \rightarrow q \]
\[ \therefore \neg p \]

Tautology:
\[ (\neg q \land (p \rightarrow q)) \rightarrow \neg p \]

Inference Rule:
Modus Tollens

Modus Tollens is Latin for "the way that denies by denying"

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I am not a great skater.

∴ I am not a great skater or I am a monkey.

\[ p \]
\[ \therefore p \lor q \]

Tautology:
\[ p \rightarrow (p \lor q) \]

Inference Rule:
Addition or Weakening
**Proofs - Simplification**

I am not a great skater and you are sleepy.

∴ you are sleepy.

\[
p \land q \\ (p \land q) \rightarrow p
\]

**Proofs - Disjunctive Syllogism**

I am a great eater or I am a great skater.
I am not a great skater.

∴ I am a great eater!

\[
p \lor q \\ \neg q \\ ((p \lor q) \land \neg q) \rightarrow p
\]
Problems - Hypothetical Syllogism

If you are an athlete, you are always hungry.
If you are always hungry, you have a snickers in your backpack.

∴ If you are an athlete, you have a snickers in your backpack.

$p \rightarrow q$
$q \rightarrow r$

$\therefore p \rightarrow r$

Tautology:

$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Inference Rule:
Hypothetical Syllogism

Problems - Conjunction

I am a great athlete
I am always hungry

∴ I am a great athlete and always hungry

$p$
$q$

$\therefore p \land q$

Tautology:

$[(p \land q) \rightarrow (p \land q)]$

Inference Rule:
Conjunction
Problems - fallacies

Rules of inference, appropriately applied give valid arguments.

Mistakes in applying rules of inference are called fallacies.

Valid Argument or Fallacy?

If I am Bonnie Blair, then I skate fast
I skate fast!

∴ I am Bonnie Blair

I am Bonnie Blair

I’m Eric Heiden

If you don’t give me $10, I bite your ear.
I bite your ear!

∴ You didn’t give me $10.

I’m just mean.

Affirming the conclusion.

Not a tautology.
Valid Argument or Fallacy?

If it rains then it is cloudy.
It does not rain.

\[
\therefore \text{It is not cloudy}
\]

If it is a car, then it has 4 wheels.
It is not a car.

\[
\therefore \text{It doesn’t have 4 wheels.}
\]

Denying the hypothesis.

\[((p \rightarrow q) \land \lnot p) \rightarrow \lnot q\]
Not a tautology.

ATV

Proof Techniques – direct proofs

Here’s what you know:
Pat is a math major or a CS major.
If Pat does not like discrete math, Pat is not a CS major.
If Pat likes discrete math, Pat is smart.
Pat is not a math major.

Can you conclude Pat is smart?

\[(M \lor C) \land (\lnot D \rightarrow \lnot C) \land (D \rightarrow S) \land (\lnot M) \rightarrow S?\]
Proof Techniques - direct proofs

In general, to prove \( p \rightarrow q \), assume \( p \) and show that \( q \) follows.

\[ ((M \lor C) \land (\neg D \rightarrow \neg C) \land (D \rightarrow S) \land (\neg M)) \rightarrow S? \]

Proof Techniques - direct proofs

1. \( M \lor C \) Given
2. \( \neg D \rightarrow \neg C \) Given
3. \( D \rightarrow S \) Given
4. \( \neg M \) Given
5. \( C \) DS (1,4)
6. \( D \) MT (2,5)
7. \( S \) MP (3,6)

Pat is smart!
Proof Techniques - direct proofs

A totally different example:
Prove that if $n$ is odd, then $5n + 3$ is even.
Before we prove it, we need to define even and odd.
How can we define an even number?

The integer $n$ is even if there exists an integer $k$ such that $n = 2k$

How can we define an odd number?

The integer $n$ is odd if there exists an integer $k$ such that $n = 2k + 1$

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Proof that if $n$ is odd, then $5n + 3$ is even.

Suppose $n$ is odd,
Therefore $n = 2k + 1$ for some integer $k$.

Therefore $5n + 3 = 5(2k + 1) + 3$
= $10k + 5 + 3 = 10k + 8$
= $2(5k + 4)$ or $2(k')$
Example:
Prove that if \( n \) is odd, then \( n^2 \) is odd.