Announcements

- Quiz 2 is today
- Exam 1 next week!

Set Theory (Review)

A set is an unordered collection of elements.

Some examples:

- \{1, 2, 3\} is the set containing “1” and “2” and “3.”
- \{1, 1, 2, 3, 3\} = \{1, 2, 3\} since repetition is irrelevant.
- \{1, 2, 3\} = \{3, 2, 1\} since sets are unordered.
- \{1, 2, 3, …\} is a way we denote an infinite set (in this case, the natural numbers).
- \varnothing = {} is the empty set, or the set containing no elements.

Note: \varnothing \neq \{\varnothing\}
Set Theory - Definitions and notation

\[ x \in S \text{ means } \text{“} x \text{ is an element of set } S. \text{”} \]
\[ x \notin S \text{ means } \text{“} x \text{ is not an element of set } S. \text{”} \]

\[ A \subseteq B \text{ means } \text{“} A \text{ is a subset of } B. \text{”} \]

or, “\( B \) contains \( A \).”
or, “every element of \( A \) is also in \( B \).”
or, \( \forall x ((x \in A) \rightarrow (x \in B)). \)

\[ \begin{array}{c}
A \\
\subseteq \\
B
\end{array} \]

Venn Diagram

Set Theory - Definitions and notation

\[ A \subseteq B \text{ means } \text{“} A \text{ is a subset of } B. \text{”} \]

\[ A = B \text{ if and only if } A \text{ and } B \text{ have exactly the same elements.} \]

iff, \( A \subseteq B \) and \( B \subseteq A \)
iff, \( \forall x ((x \in A) \leftrightarrow (x \in B)). \)

So to show equality of sets \( A \) and \( B \), show:
\[ A \subseteq B \]
\[ B \subseteq A \]
Set Theory - Definitions and notation

A \subset B means “A is a proper subset of B.”

- A \subseteq B, and A \neq B.
- \forall x ((x \in A) \rightarrow (x \in B)) \land \neg \forall x ((x \in B) \rightarrow (x \in A))
- \forall x ((x \in A) \rightarrow (x \in B)) \land \exists x ((x \in B) \land \neg (x \in A))

Quick examples:
{1,2,3} \subseteq {1,2,3,4,5}
{1,2,3} \subset {1,2,3,4,5}

Is \emptyset \subseteq \{1,2,3\}?
Yes! \forall x (x \in \emptyset) \rightarrow (x \in \{1,2,3\}) holds, because (x \in \emptyset) is false.

Is \emptyset \in \{1,2,3\}? No!
Is \emptyset \subseteq \{\emptyset,1,2,3\}? Yes!
Is \emptyset \in \{\emptyset,1,2,3\}? Yes!
Is \emptyset \subseteq \{\emptyset,1,2,3\}? Yes!
### Set Theory - Definitions and notation

#### Quiz time:

<table>
<thead>
<tr>
<th>Question</th>
<th>Answer</th>
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<tbody>
<tr>
<td>Is ( x \subseteq {x} )?</td>
<td>No</td>
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### Set Theory - Ways to define sets

- **Explicitly:** \( \{\text{John, Paul, George, Ringo}\} \)
- **Implicitly:** \( \{1,2,3,\ldots\} \), or \( \{2,3,5,7,11,13,17,\ldots\} \)
- **Set builder:** \( \{x : x \text{ is prime}\} \), \( \{x | x \text{ is odd}\} \).
  
  In general \( \{x : P(x) \text{ is true}\} \), where \( P(x) \) is some description of the set.

: and | are read “such that” or “where”

Ex. Let \( D(x,y) \) denote “\( x \) is divisible by \( y \)”

Give another name for
\[
\{ x : \forall y ((y > 1) \land (y < x)) \rightarrow \neg D(x,y) \}.
\]

What is this set of numbers?

Primes
Set Theory - Cardinality

If $S$ is finite, then the cardinality of $S$, $|S|$, is the number of distinct elements in $S$.

If $S = \{1,2,3\}$, \hspace{1cm} |S| = 3
If $S = \{3,3,3,3,3\}$, \hspace{1cm} |S| = 1
If $S = \emptyset$, \hspace{1cm} |S| = 0
If $S = \{1, \{1\}, \{1,\{1\}\}\}$, \hspace{1cm} |S| = 3

If $S = \{0,1,2,3,\ldots\}$, $|S|$ is (one kind of) infinity. (more on this later)

Set Theory - Power sets

If $S$ is a set, then the power set of $S$ is $2^S = \{ x : x \subseteq S \}$.

If $S = \{a\}$, \hspace{1cm} $2^S = \{\{\}, \{a\}\}$. \hspace{1cm} aka $P(S)$
If $S = \{a,b\}$, \hspace{1cm} $2^S = \{\{\}, \{a\}, \{b\}, \{a,b\}\}$.
If $S = \emptyset$, \hspace{1cm} $2^S = \{\}$. \hspace{1cm} We say, “$P(S)$ is the set of all subsets of $S$.”
If $S = \{\emptyset,\{\emptyset\}\}$, \hspace{1cm} $2^S = \{\{\}, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset,\{\emptyset\}\}\}$.

Fact: if $S$ is finite, $|P(S)| = 2^{|S|}$. (if $|S| = n$, $|P(S)| = 2^n$)
### Set Theory - Cartesian Product

The **Cartesian Product** of two sets $A$ and $B$ is:

$$ A \times B = \{ <a, b> : a \in A \land b \in B \} $$

If $A = \{ \text{Charlie, Lucy, Linus} \}$, and
$B = \{ \text{Brown, VanPelt} \}$, then

$$ A \times B = \{ <\text{Charlie, Brown}>, <\text{Charlie, VanPelt}>, <\text{Lucy, Brown}>, >, <\text{Lucy, VanPelt}>, <\text{Linus, Brown}>, <\text{Linus, VanPelt}> \} $$

$$ A_1 \times A_2 \times ... \times A_n = \{ a_1, a_2, ..., a_n \} : a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n \} $$

### Set Theory - Operators

The **union** of two sets $A$ and $B$ is:

$$ A \cup B = \{ x : x \in A \lor x \in B \} $$

If $A = \{ \text{Charlie, Lucy, Linus} \}$, and
$B = \{ \text{Lucy, Desi} \}$, then

$$ A \cup B = \{ \text{Charlie, Lucy, Linus, Desi} \} $$
Set Theory - Operators

The intersection of two sets $A$ and $B$ is:
$$A \cap B = \{ x : x \in A \land x \in B \}$$

If $A = \{\text{Charlie, Lucy, Linus}\}$, and
$B = \{\text{Lucy, Desi}\}$, then
$$A \cap B = \{\text{Lucy}\}$$

Another example

If $A = \{x : x \text{ is a US president}\}$, and
$B = \{x : x \text{ is deceased}\}$, then
$$A \cap B = \{x : x \text{ is a deceased US president}\}$$
## Set Theory - Operators

### One more example

If \( A = \{x : x \text{ is a US president}\} \), and
\( B = \{x : x \text{ is in this room}\} \), then

\[
A \cap B = \{x : x \text{ is a US president in this room}\} = \emptyset
\]

Sets whose intersection is empty are called **disjoint** sets.

### Set Theory - Operators

The **complement** of a set \( A \) is:

\[
\overline{A} = \{x : x \notin A\}
\]

If \( A = \{x : x \text{ is bored}\} \), then \( \overline{A} = \{x : x \text{ is not bored}\} \)

\( \emptyset = U \)
and
\( U = \emptyset \)
Set Theory - Operators

The set difference, $A - B$, is (also written $A \setminus B$):

\[
A - B = \{ x : x \in A \land x \notin B \}
\]

\[
A - B = A \cap \overline{B}
\]

Set Representation

• How could you represent a set on a computer?
  – Bitmap Representation
    • The set of 10 numbers, with 1, 3, and 5 set
      – 1010100000
  – Linked List

1 → 3 → 5

• How would you complement the set?
  – Bitmap?
  – Linked List?
Set Theory - Operators

The **symmetric difference**, \( A \oplus B \), is:
\[
A \oplus B = \{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \}
\]
\[
= (A - B) \cup (B - A)
\]

Proof:
\[
\{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \}
\]
\[
= \{ x : (x \in A - B) \lor (x \in B - A) \}
\]
\[
= \{ x : x \in ((A - B) \cup (B - A)) \}
\]
\[
= (A - B) \cup (B - A)
\]
### Set Theory - Famous Identities

#### Identity
- $A \cap U = A$
- $A \cup \emptyset = A$

#### Domination
- $A \cup U = U$
- $A \cap \emptyset = \emptyset$

#### Idempotent
- $A \cup A = A$
- $A \cap A = A$

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### Set Theory - Famous Identities

- **Excluded Middle**
  \[ A \cup \overline{A} = U \]
- **Uniqueness**
  \[ A \cap \overline{A} = \emptyset \]
- **Double complement**
  \[ \overline{\overline{A}} = A \]
Set Theory - Famous Identities

- **Commutativity**
  \[ A \cup B = B \cup A \]
  \[ A \cap B = B \cap A \]

- **Associativity**
  \[ (A \cup B) \cup C = A \cup (B \cup C) \]
  \[ (A \cap B) \cap C = A \cap (B \cap C) \]

- **Distributivity**
  \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

- **DeMorgan’s I**
  \[ \overline{A \cup B} = \overline{A} \cap \overline{B} \]

- **DeMorgan’s II**
  \[ \overline{A \cap B} = \overline{A} \cup \overline{B} \]

Hand waving is good for intuition, but we aim for a more formal proof.
Set Theory – 4 Ways to prove identities

- Show that $A \subseteq B$ and that $B \subseteq A$.

- Use a membership table. Like truth tables

- Use previously proven identities. Like $=$

- Use logical equivalences to prove equivalent set definitions. Not hard, a little tedious

Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

1. $(\subseteq)$ $(x \in A \cup B) \rightarrow (x \notin A \cup B) \rightarrow (x \notin A \text{ and } x \notin B) \rightarrow (x \in \overline{A} \cap \overline{B})$

2. $(\supseteq)$ $(x \in \overline{A} \cap \overline{B}) \rightarrow (x \notin A \text{ and } x \notin B)$
   \[ \rightarrow (x \notin A \cup B) \rightarrow (x \in A \cup B) \]
Set Theory – 4 Ways to prove identities

Prove that \((\overline{A \cup B}) = \overline{A} \cap \overline{B}\) using a membership table.

0 : x is not in the specified set
1 : otherwise

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Haven’t we seen this before?

Set Theory – 4 Ways to prove identities

Prove that \((\overline{A \cup B}) = \overline{A} \cap \overline{B}\) using logically equivalent set definitions.

\[ (\overline{A \cup B}) = \{ x : \neg (x \in A \lor x \in B) \} \]

= \{ x : \neg (x \in A) \land \neg (x \in B) \} \\
= \{ x : (x \in \overline{A}) \land (x \in \overline{B}) \} \\
= \overline{A} \cap \overline{B} \]
Set Theory - Generalized Union

\[ \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \]

Ex. Let \( U = \mathbb{N} \), and define:

\[ A_i = \{ x : \exists k > 1, x = ki, k \in \mathbb{N} \} \]

\( A_1 = \{2,3,4,\ldots\} \)
\( A_2 = \{4,6,8,\ldots\} \)
\( A_3 = \{6,9,12,\ldots\} \)

Then

\[ \bigcup_{i=2}^{\infty} A_i = ? \]

a) Primes
b) Composites
c) \( \emptyset \)
d) \( \mathbb{N} \)
e) I have no clue.
Set Theory - Generalized Intersection

\[
\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n
\]

Ex. Let \( U = \mathbb{N} \), and define:

\[
A_i = \{ x : \exists k, x = ki, k \in \mathbb{N} \}
\]

\( A_1 = \{1,2,3,4,...\} \)

\( A_2 = \{2,4,6,...\} \)

\( A_3 = \{3,6,9,...\} \)

Then

\[
\bigcap_{i=1}^{n} A_i = \text{Multiples of LCM}(1,..,n)
\]
Set Theory - Inclusion/Exclusion

Example:
How many people are wearing a watch?
How many people are wearing sneakers?

How many people are wearing a watch OR sneakers?

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

Example:
There are 83 cs majors.
40 are taking cs240.
31 are taking cs101.
22 are taking both.

How many are taking neither?

\[ 83 - (40 + 31 - 22) = 34 \]
Set Theory - Generalized Inclusion/Exclusion

Suppose we have:

And I want to know $|A \cup B \cup C|$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$