In questions 1 – 4 below mark each statement TRUE or FALSE. Assume that the statement applies to all sets. (1 pt each)

1. \( A-(B-C)=(A-B)-C \)
   False

2. \( A \cap (B \cup C)=(A \cup B) \cap (A \cup C) \)
   False

3. If \( A \cap C=B \cap C \), then \( A=B \)
   False

4. If \( A \cap B=A \cup B \), then \( A=B \)
   True

5. [4pts] A computer network consists of six computers. Each computer is directly connected to zero or more of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of computers.
   Ans: Each computer can be connected to 0,1,2,3,4, or 5 other computers, but it is not possible in the network to have a computer connected to 0 others and a computer connected to all 5 others. Therefore there are only five possible connection numbers, which is smaller than the number of computers. By the Pigeonhole Principle at least two must have the same number of connections.
6. (2 pts) Suppose \( U = \{1, 2, ..., 9\}, A = \) all multiples of 2, \( B = \) all multiples of 3, and \( C = \{3, 4, 5, 6, 7\} \). Find \( C - (B - A) \).

Ans: \{4,5,6,7\}.

7. (2 pts) How many permutations of the seven letters A, B, C, D, E, F, G have the two vowels before the five consonants?

Ans: \( 2 \times 5! \)

8. (2 pts) Consider a club with 20 women and 17 men, suppose it needs to form a committee of size six. How many committees are possible if the committee must have at least two men?

Ans: \( \binom{17}{2} \binom{20}{4} + \binom{17}{3} \binom{20}{3} + \binom{17}{4} \binom{20}{2} + \binom{17}{5} \binom{20}{1} + \binom{17}{6} \binom{20}{0} \).

In questions 9-12 find the “best” big-O notation to describe the complexity of the algorithm. (1 pt each). Choose your answers from the following: 1, \( \log_2 n \), \( n \), \( n \log_2 n \), \( n^2 \), \( n^3 \), ..., \( 2^n \), \( n! \).

9. A binary search of \( n \) elements.

Ans: \( \log_2 n \)

10. A linear search to find the smallest number in a list of \( n \) numbers.

Ans: \( n \)

11. An algorithm that lists all ways to put the numbers 1,2,3,...,\( n \) in a row.

Ans: \( n! \)

12. An algorithm that prints all bit strings of length \( n \).

Ans: \( 2^n \)

In questions 13 and 14 below state whether the rule describes a function with the given domain and codomain. (1 pt each)

13. \( F: \mathbb{R} \rightarrow \mathbb{R} \) where \( f(x)= x^2 \), if \( x \leq 2 \) and \( f(x) = x-1 \), if \( x \geq 4 \)

Ans: Not a function \( f(3) \) undefined

14. \( h: \mathbb{R} \rightarrow \mathbb{R} \) where \( h(x)=\sqrt{x} \)

Ans: Not a function in \( \mathbb{R} \) for \( x<0 \)
15. (1 pt) Suppose \( f: \mathbb{N} \rightarrow \mathbb{N} \) has the rule \( f(n) = 3n^2 - 1 \). Determine whether \( f \) is 1-1.

Ans: Yes

16. (1 pt) Suppose \( f: \mathbb{N} \rightarrow \mathbb{N} \) has the rule \( f(n) = 4n^2 + 1 \). Determine whether \( f \) is onto \( \mathbb{N} \).

Ans: No

17. (2 pts) Suppose \( g:A \rightarrow B \) and \( f:B \rightarrow C \) where \( A=\{a,b,c,d\} \), \( B=\{1,2,3\} \), \( C=\{2,3,6,8\} \) and \( f \) and \( g \) are defined by \( g=\{(a,2),(b,1),(c,3),(d,2)\} \) and \( f=\{(1,8),(2,3),(3,2)\} \). Find \( f \circ g \).

Ans: \( \{(a,3),(b,8),(c,2),(d,3)\} \).

18. (2 pts) Find a formula that generates the following sequence: 1,1/3,1/5,1/7,1/9,..

\[ a_n = \frac{1}{2n-1} \]

19. (2 pts) Find the following:

\[ \sum_{j=1}^{3} \sum_{i=1}^{j} ij \]

Ans: 25.

20. (2 pts) Rewrite \( \sum_{i=3}^{4} (i^2 + 1) \) so that the index of summation has lower limit 0 and upper limit 7.

Ans: \( \sum_{i=0}^{7} (i - 3)^2 + 1 \).
In question 21 below give a recursive definition (with initial condition(s)) of \( \{a_n\} \) \( n=1,2,3,... \). (2 pts)

21. The sequence \( a_1=16, a_2=13, a_3=10, a_4=7, \ldots \)

Ans: \( a_n=a_{n-1}-3, a_1=16 \).

22. (4 pts) Suppose that the only currency were 4-dollar bills and 5-dollar bills. Show that any dollar amount greater than 11 dollars could be made from a combination of these bills.

Strong Induction Solution:
Base Case: \( n = 12, n = 13, n = 14, n = 15 \)
We can form $12 using 3, 4-dollar bills
We can form $13 using 2, 4-dollar bills and 1 5-dollar bills
We can form $14 using 1, 4-dollar bills and 2 5-dollar bills
We can form $15 using 3, 5-dollar bills

Induction Step
Let \( n \geq 15 \)
Assume \( P(k) \) is true for \( 12 \leq k \leq n \), that is dollar amounts of \( k \) dollars can be formed with 4-dollar bills and 5-dollar bills (Inductive Hypothesis)
Prove \( P(n+1) \)
To form amount of \( n+1 \) dollars, use the dollar bills that form the correct amount of \( n-3 \) dollars (from I.H) with a 4-dollar bill

Could also use “regular” mathematical induction

23. (4 pts) Prove or disprove: if \( A, B, \) and \( C \) are sets, then \( A - (B \cap C) = (A - B) \cup (A - C) \)

True, since \( A - (B \cap C) \)

\[ = A \cap \overline{(B \cap C)} \] by def of set difference

\[ = A \cap (B \cup \overline{C}) \] by de Morgans

\[ = (A \cap B) \cup (A \cap \overline{C}) \] by Distribution

\[ = (A - B) \cup (A - C) \] by def of set difference